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Integer-valued and almost integer-valued functions

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Abstract

In this article, we discuss conditions so that complex entire functions are integer-valued, by means of methods based on Diophantine problems. We also describe how are deduced conditions to be "almost" integer-valued.

Keywords: Integer-valued functions, Almost integer-valued functions, Lattice, Geometry of numbers, Transcendence method.

1 Introduction

We first consider a naive question as follows. Denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let $F(z)$ be a complex function in one variable which satisfies $F(u) \in \overline{\mathbb{Q}}$ for any $u \in \overline{\mathbb{Q}}$.

Could we specify any properties for the function $F(z)$?

We know that such $F(z)$ is NOT necessarily algebraic function ; we had examples in 1894 due to P. Stäckel of a transcendental function taking algebraic values at all algebraic points. In fact, Stäckel's showed a more general statement : *for any countable set S, and for any set T being dense in the complex plane, there exists an entire function $F(z) = \sum_{n=0}^{\infty} f_n z^n$ with rational f_n, satisfying $F(u) \in T$ for any $u \in S$.* Thus we have $F(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$ for $S = T = \overline{\mathbb{Q}}$. Moreover he constructed in 1902 a transcendental function $F(z) = \sum_{n=0}^{\infty} f_n z^n$ with rational f_n, analytic in a neighbourfood of the origin, with the property that the both $F(z)$ and its inverse function take algebraic values at all algebraic points$\in \overline{\mathbb{Q}}$ in the neighbourfood.

We also recall that the Hermite-Lindemann theorem shows $\exp(\alpha) \notin \overline{\mathbb{Q}}$ for any $\alpha \in \overline{\mathbb{Q}}$, $\alpha \neq 0$, which says, such transcendental function always takes transcendental values at any non-trivial algebraic point. Stäckel's result notices that this is not true for all transcendental functions. F. Beukers and J. Wolfart gave in 1988 a condition for the algebraicity of the values of Gauss' hypergeometric function $F(z)$ and Wolfart made a criterion to distinguish whether a Gauss' hypergeometric function takes algebraic values at algebraic points or not [Be-Wo] : *if Gauss' hypergeometric function* $F(z)$ *is algebraic over* $\mathbb{C}(z)$, *then* $F(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$, *otherwise there are hypergeometric functions, either* $F(z)$ *with* $F(\xi) \in \overline{\mathbb{Q}}$ *for only finitely many* $\xi \in \overline{\mathbb{Q}}$, *or,* $F(z)$ *such that there is a subset* $E \subset \overline{\mathbb{Q}}$ *which is dence in* $\mathbb{C}$ *satisfying* $F(\xi) \in \overline{\mathbb{Q}}$ *whenever* $\xi \in E$.

On the other hand, there are many examples of transcendental functions such that $F(\mathbb{N}) \subset \mathbb{Z}$, or $F(\mathbb{Q}) \subset \overline{\mathbb{Q}}$ *e. g.* $F(z) = 2^z$.

Now we ask, what it is, the function $F(z)$ with $F(\mathbb{N}) \subset \mathbb{Z}$ or $F(\mathbb{Z}) \subset \mathbb{Z}$. We have the following fundamental result due to G. Pólya [Po] in 1915 (*see also* [Ha]).

Definition 1 *Let* $F(z)$ *be an entire function in* $\mathbb{C}$. *Write* $|F|_r = \sup_{|z| \leq r} |F(z)|$ *and define* $\tau(F)$ *the order of exponential type of* $F(z)$; $\tau(F) = \limsup_{r \to +\infty} \frac{\log |F|_r}{r}$.

Theorem A (Pólya) *Let* $F(z)$ *be an entire function in* $\mathbb{C}$ *with* $F(\mathbb{N}) \subset \mathbb{Z}$. *Suppose* $\tau(F) < \log 2$. *Then* $F(z)$ *is a polynomial.*

Pólya considered also the case $F(\mathbb{Z}) \subset \mathbb{Z}$ (*see also* [Ca]).

Theorem B (Pólya) *Let* $F(z)$ *be an entire function in* $\mathbb{C}$ *with* $F(\mathbb{Z}) \subset \mathbb{Z}$. *Suppose* $\tau(F) < \log(\frac{3+\sqrt{5}}{2})$. *Then* $F(z)$ *is a polynomial.*

In Theorem A and B, we see that $F(z)$ is necessarily a polynomial in $\mathbb{Q}[z]$, but not in $\mathbb{Z}[z]$ (consider for example, $\frac{1}{2}z(z+1)$).

The bounds $\log 2$ and $\log(\frac{3+\sqrt{5}}{2})$ are optimal because of 2^z and $\left(\frac{3+\sqrt{5}}{2}\right)^z + \left(\frac{3-\sqrt{5}}{2}\right)^z$.

2 Results

We see, $F(z)$ is a polynomial with coefficients in $\mathbb{Q}$, which is equivalent to say the functions $z^h F(z)^k$ $(h, k \in \mathbb{N} \cup \{0\})$ are linearly dependent over $\mathbb{Q}$. Then a natural generalization of works of Pólya is to seek a sufficient condition such that several functions $f_1, \cdots, f_L$ are linearly dependent over $\mathbb{Q}$.

For $\zeta_1, \zeta_2, \cdots \in \mathbb{C}$ denote by $r(N) := \max_{1 \leq n \leq N} |\zeta_n|$.

We then show

Theorem 1 *Let L and N_0 be rational integers with $1 < N_0 < L$. There are constants $C_1 > 0$ and $C_2 > 0$ depending only on L, N_0 satisfying the following. Let $\zeta_1, \zeta_2, \cdots \in \mathbb{C}$ be infinite complex points pairwise distinct. Let $f_1, \cdots, f_L$ be entire functions in $\mathbb{C}$. Suppose $f_j(\zeta_n) \in \mathbb{Z}$ for any j, n with $1 \leq j \leq L$ and $n \geq 1$. If we have $\max_{1\leq j\leq L} \log |f_j|_{C_1 r(N)} \leq C_2 N$ for any $N \geq N_0$, then the functions $f_1, \cdots, f_L$ are linearly dependent over $\mathbb{Q}$.*

We are able to calculate $C_1 > 0$ and $C_2 > 0$ in an explicit manner. Several consequences of Theorem 1 are, for instance, as follows :

Corollary 2 *Let $F(z)$ be an entire function in $\mathbb{C}$ with $F(\mathbb{N}) \subset \mathbb{Z}$. Suppose $\tau(F) \leq \frac{1}{40}$. Then $F(z)$ is a polynomial over $\mathbb{Q}$.*

Proof We consider $f_1, \cdots, f_L$ as $\dfrac{z(z-1)\cdots(z-h+1)}{h!} \cdot F(z)^k$ $(h, k \in \mathbb{N} \cup \{0\})$. □

Corollary 3 *Let F be an entire function with $F(\mathbb{N}) \subset \mathbb{Z}$ with $\tau(F) \leq \frac{11}{14}$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that F satisfies the functional equation*

$$\sum_{h=0}^{N_1} \sum_{k=0}^{N_2} a_{hk} z^h F(z+k) = 0$$

with $a_{hk} \in \mathbb{Q}$ not all zero.

3 Almost integer-valued functions

Let us try to relax our condition in Theorem 1. We still want as consequence the functions to be polynomials over $\mathbb{Q}$ or linearly dependent over $\mathbb{Q}$, however we DOUBT if we really need the functions are integer-valued in the sufficient condition. Indeed we succeeded in proving the following.

Put $\delta(N) := \min_{1\leq h<k\leq N} |\zeta_h - \zeta_k|$ for $N \geq 2$ and $\zeta_1, \zeta_2, \cdots \zeta_N \in \mathbb{C}$ distinct.
Set also $\| z \| := \min_{m\in\mathbb{Z}} |z - m|$.

Theorem 4 *Let L and N_0 be rational integers with $1 < N_0 < L$. There are constants $C_1 > 19$, $C_2 > 0$ and $C_3 > 0$ depending only on L, N_0 satisfying the following. Let $\zeta_1, \zeta_2, \cdots \in \mathbb{C}$ be infinite complex points pairwise distinct. Let $f_1, \cdots, f_L$ be entire functions in $\mathbb{C}$. Assume*

$$\max_{1\leq j\leq L} \log |f_j|_{C_1 r(N)} \leq C_2 N$$

for any $N \geq N_0, 1 \leq j \leq L$. For $1 \leq \forall j \leq L$, $\forall n \geq 1$ assume also

$$\| f_j(\zeta_n) \| \leq C_3 e^{-3n} n^{\frac{n}{20}} \left(\frac{\delta(2n+1)}{r(2n+1)} \right)^{n+1}.$$

Then the functions $f_1, \cdots, f_L$ are linearly dependent over $\mathbb{Q}$.

We may explicitly calculate C_1, C_2 and C_3. We present consequences of Theorem 4 :

Corollary 5 *Let $F(z)$ be an entire function in $\mathbb{C}$. Suppose $\tau(F) \leq \frac{1}{451}$ and $\| F(n) \| \leq e^{-5n}$ for any $n \in \mathbb{N}$. Then $F(z)$ is a polynomial over $\mathbb{Q}$.*

4 Outline of the proof of theorems

We prove the theorems by Schneider's method which is one of main transcendence methods to deal with Diophantine problems.

Proof of Theorem 1

[First step] : construction of an auxiliary function
We may suppose that each $f_j(z)$, $(1 \leq j \leq L)$ is not identically zero. Then we know that the zeroes of the functions are isolated. We start to construct the function $F(z) := \sum_{j=1}^{L} p_j f_j(z)$ such that the coefficients $p_j \in \mathbb{Z}$ $(1 \leq j \leq L)$ are not all zero and that, for any n, $1 \leq n \leq N_0$, $F(\zeta_n) = 0$. This requires to solve a system in unknowns $\in \mathbb{Z}$ by Siegel's Lemma (the algebraic one, *see* [Da-Hi] and [Hi]). We use the Lemma to conclude that there exist $p_j \in \mathbb{Z}$ $(1 \leq j \leq L)$ not all zero such that

$$\max_{1 \leq j \leq L} \log |p_j| \leq \frac{1}{L - N_0} \log L + \frac{N_0(N_0 - 1)}{L} \log \frac{C_1}{2}.$$

[Second step] : extrapolation

We consider for each $N \geq N_0$ the following properties :

$A(N) : F(\zeta_n) = 0$ for $1 \leq \forall n \leq N$
$B(N) : |F|_{r(N+1)} < 1$.

We shall show $\quad A(N) \Longrightarrow B(N)$ and $B(N) \Longrightarrow A(N+1)$.

[Proof of $A(N) \Longrightarrow B(N)$] :
By the hypothesis $A(N)$, we see that the function F has at least N zeroes in the set $\{z \in \mathbb{C} : |z| \leq r(N+1)\}$. The Lemma of Schwarz (in fact the residue formula ; *see* [Gr] and [Gr-Mi-Wa]) shows

$$\log |F|_{r(N+1)} \leq \log |F|_{C_1 r(N+1)} - N \log \frac{C}{2}$$

then we have

$$\log |F|_{r(N+1)} \leq \frac{1}{L - N_0} \log L - \frac{N_0}{L} \log \frac{C_1}{2}.$$

The suitable choice of C_1 allows us conclude.

[Proof of $B(N) \Longrightarrow A(N+1)$] :
The property $B(N)$ implies $|F(\zeta_n)| < 1$ for any $1 \leq n \leq N+1$. Since $f_j(\zeta_n) \in \mathbb{Z}$ then we obtain $F(\zeta_n) \in \mathbb{Z}$ namely $F(\zeta_n) = 0$.

[Third step] : conclusion
The property $A(N_0)$ holds by the construction of F in the First step. Therefore by the Second and the Third steps we see that $A(N)$ and $B(N)$ are true for any $N \geq N_0$. If F is not identically zero, the zeroes are isolated, which implies $r(N) \to \infty$, thus by Liouville's theorem $|F|_{r(N)} \to \infty$ for $N \to \infty$, that contradicts with $B(N)$. Then F is identically zero. □

Proof of Theorem 4

[First step] : construction of an auxiliary function
We choose the constants C_1, C_2 so as to satisfy the both :

$$\frac{1}{4}\log\frac{C_1-1}{2} - \frac{1}{2}\log 3 - \frac{L}{L-N_0}\log L - \frac{N_0}{2(L-N_0)} - \log\frac{C_1}{C_1-1} > 0,$$

$$C_2 N \leq \frac{(L-N_0)(N-1)}{L}\left(\frac{1}{2}\log\frac{C_1-1}{2} - \log 3\right) - \log L - \frac{N_0}{2L} - \frac{L-N_0}{L}\log\frac{C_1}{C_1-1}.$$

By hypothesis, there are integers a_{jn} such that for each $1 \leq j \leq L, 1 \leq n \leq N_0$:

$$\| f_j(\zeta_n) \| = |f_j(\zeta_n) - a_{jn}|.$$

The obvious inequality

$$\log|a_{jn}| \leq \log\left(\frac{1}{2} + |f_j(\zeta_n)|\right) \leq \max\left(\frac{1}{2} + \log|f_j(\zeta_n)|, \quad \log\frac{3}{2}\right)$$

and the assumption of Theorem 4 give us for any $1 \leq j \leq L, 1 \leq n \leq N_0$:

$$\log|a_{jn}| \leq \frac{1}{2} + C_2 N_0.$$

We construct the function $F(z) := \sum_{j=1}^{L} p_j f_j(z)$ such that the coefficients $p_j \in \mathbb{Z}$ $(1 \leq j \leq L)$ are not all zero and for any n, $\frac{N_0}{2} \leq n \leq N_0$ that

$$\sum_{j=1}^{L} p_j a_{jn} = 0.$$

Since $L > [N_0/2 + 1]$ we may solve the system again by Siegel's Lemma to get $p_j \in \mathbb{Z}$ $(1 \leq j \leq L)$ not all zero with

$$\max_{1 \leq j \leq L} \log|p_j| \leq \frac{N_0}{L-N_0}\left(\log L + \frac{1}{2} + C_2 N_0\right).$$

We then have

$$\log|F|_{C_1 r(N)} \leq \frac{L}{L-N_0}\left(\log L + C_2 N + \frac{N_0}{2L}\right).$$

[Second step] : extrapolation
We consider for each $N \geq N_0$ the following properties :

$A(N) : \sum_{j=1}^{L} p_j a_{jn} = 0$ for $\frac{N}{2} \leq n \leq N$
$B(N) : |F|_{r(N+1)} < 3^{-N} + 3^{-1}$.

We shall show that $\quad A(N) \Longrightarrow B(N)$ and $B(N) \Longrightarrow A(N+1)$.

[Proof of $A(N) \Longrightarrow B(N)$] : Put

$$C_3 = e^{-3}\left(L^{\frac{L}{L-N_0}} \cdot \exp\left(\frac{N_0}{L-N_0}\left(\frac{1}{2} + C_2 N_0\right)\right)\right)^{-1}.$$

By the hypothesis $A(N)$, we see

$$|F(\zeta_n)| = |\sum_{j=1}^{L} p_j(f_j(\zeta_n) - a_{jn})| \leq C_3^{-1} e^{-3} \max_{1\leq j\leq L} \| f_j(\zeta_n) \|$$

which shows under the assumption of $\| f_j(\zeta_n) \|$:

$$|F(\zeta_n)| \leq n^{\frac{n}{20}} \left(\frac{\delta(2n+1)}{e^3 r(2n+1)}\right)^{n+1}.$$

Because of

$$\frac{\delta(2n+1)}{e^3 r(2n+1)} < 1, \quad \frac{\delta(n)}{r(n)} \geq \frac{\delta(n+1)}{r(n+1)}$$

we have

$$\max_{\frac{N}{2}\leq n\leq N} |F(\zeta_n)| \leq e^{-3-\frac{3N}{2}} N^{\frac{N}{20}} \frac{\delta(N+1)}{r(N+1)}^{\frac{N}{2}+1}.$$

On the other hand, use the inequality from the residue formula :

Let f be a function analytic in $|z| \leq R$ in $\mathbb{C}$ and be $\zeta_0, \zeta_1, \cdots \zeta_l \in \mathbb{C}$ in $|z| \leq R$. Then we have

$$|f(\zeta_0)| \leq E_1 + E_2$$

where

$$E_1 = |f|_R \cdot \frac{R}{R - |\zeta_0|} \prod_{n=1}^{l} \frac{|\zeta_0 - \zeta_n|}{R - |\zeta_n|},$$

$$E_2 = \prod_{k=1}^{l} |\zeta_0 - \zeta_k| \sum_{n=1}^{l} \left(\frac{|f(\zeta_n)|}{|\zeta_0 - \zeta_n|} \cdot \prod_{i=1, i\neq n}^{l} \frac{1}{|\zeta_i - \zeta_n|}\right).$$

Take $\zeta_0 \in \mathbb{C}$ with $|\zeta_0| = r = r(N+1)$ and $\zeta_0 \neq \zeta_i (\frac{N}{2} \leq i \leq N)$. We now get thanks to the above inequality :

$$|F|_r \le |F|_R \cdot \frac{R}{R-r} \prod_{\frac{N}{2}\le n\le N} \frac{|\zeta_0-\zeta_n|}{R-|\zeta_n|} + \prod_{\frac{N}{2}\le k\le N} |\zeta_0-\zeta_k| \sum_{\frac{N}{2}\le n\le N} \left(\frac{|F(\zeta_n)|}{|\zeta_0-\zeta_n|} \cdot \prod_{i\ne n} \frac{1}{|\zeta_i-\zeta_n|} \right).$$

For $R = C_1 r$, this implies $|F|_r \le T_1 + T_2$ where

$$T_1 = |F|_R \cdot \frac{C_1}{C_1-1} \cdot \left(\frac{C_1-1}{2}\right)^{-M},$$

$$T_2 = M \max_{\frac{N}{2}\le n\le N} |F(\zeta_n)| 6^{M-1} \left(\frac{3}{M}\right)^{\frac{M}{2}} \cdot \left(\frac{r(N+1)}{\delta(N)}\right)^{M-1}$$

with $M = [N/2+1]$.

The choice of C_2 and the construction of the auxiliary function let us obtain

$$\log |T_1| \le -N \log 3.$$

Next, we see that the disc of radius $r(N) + \delta(N)/2$ contains N disjoint discs of radius $\delta(N)/2$ thus we have $N\left(\frac{\delta(N)}{2}\right)^2 \le \left(r(N) + \frac{\delta(N)}{2}\right)^2$, therefore putting $\sigma(N) := \frac{\delta(N)}{r(N)}$ we obtain $\sigma(N)\sqrt{N} \le 2 + \sigma(N)$ then

$$\sigma(N+1) \le \frac{1}{2(\sqrt{2}-1)\sqrt{N}}$$

for $N \ge 1$ and

$$\sigma(N+1) \le \frac{2\sqrt{2}}{(\sqrt{3}-1)\sqrt{N}}$$

for $N \ge 2$. The upper bound for $\max_{\frac{N}{2}\le n\le N} |F(\zeta_n)|$ gives us

$$T_2 \le M e^{-3-\frac{3M}{2}} N^{\frac{N}{20}} 6^{M-1} \left(\frac{3}{M}\right)^{\frac{M}{2}} \sigma(N+1)^{\frac{N}{2}-1-[N/2]}.$$

Consequently we have $T_2 < \frac{1}{3}$ for $N \ge 2$ and then

$$|F|_{r(N+1)} \le T_1 + T_2 \le 3^{-N} + 3^{-1}.$$

[Proof of $B(N) \Longrightarrow A(N+1)$] :
By the assumption on $\| f_j(\zeta_n) \|$ of the theorem and the property on $\sigma(N)$, we see

$$\| f_j(\zeta_{N+1}) \| \le C_3 e^{-7}$$

for any $N \ge 2$.

Using the bound for $\log |p_j|$: the coefficient of the auxiliary function, and the choice of C_3, we get

$$|\sum_{j=1}^{L} p_j a_{j,N+1}| \leq L \max_{1 \leq j \leq L} (|p_j| \parallel f_j(\zeta_{N+1}) \parallel) + |F(\zeta_{N+1})| \leq C_3^{-1} e^{-3} C_3 e^{-7} + 3^{-2} + 3^{-1} < 1$$

which implies that the integer vanishes, namely $\sum_{j=1}^{L} p_j a_{j,N+1} = 0$.

[Third step] : conclusion
Since $B(N)$ is true for any $N \geq N_0$, we have

$$|F|_{r(N+1)} < 3^{-N} + 3^{-1}$$

then Liouville's theorem assures us that F is identically zero. □

5 Higher dimensional case

Now we ask to ourselves what happens on $\mathbb{C}^m$ in several variables case, or algebraic integer-valued case. Let K be an algebaic number field of degree $[K : \mathbb{Q}] = d$ and $\mathcal{O}_K$ be the integer ring of K. In such cases, we have also a sequence of proven results. The works due to Seigo Fukasama (= Seigo Morimoto) around 1920's together with the related results of A. O. Gel'fond concerning the entire function in $\mathbb{C}$ with $F(\mathbb{Z}[i]) \subset \mathbb{Z}[i]$ consist of some fundamental concept in this area. Daihachiro Sato also investigated integer-valued functions in 1960's. F. Gramain obtained the best possible bound for the order of the entire function in $\mathbb{C}$ with $F(\mathcal{O}_K) \subset \mathcal{O}_K$ to be a polynomial when K is imaginary quadratic [Gr]. For the historical survey we refer his article. C. Pisot dealt with not only integer-valued functions but also almost integer-valued functions by interpolation method. A version in characteristic $p > 0$ is studied by M. Car and D. Adam.

Definition 2 *Let $\delta = d$ if $K \subset \mathbb{R}$ and $\delta = \frac{d}{2}$ otherwise.*

We claim that the highly important fact

$$a \in \mathbb{Z}, |a| < 1 \Longrightarrow a = 0$$

is not true in the case of algebraic integers, so we use instead, the following Size Inequality:

$$a \in \mathcal{O}_K, a \neq 0 \Longrightarrow \log |\overline{a}| \geq -(\delta - 1) \log |\overline{a}|$$

where $|\overline{a}|$ denotes the maximum of absolute values of all the conjugates of a over $\mathbb{Q}$.

Definition 3 *Let $F(z)$ be an entire function in $\mathbb{C}^m$ and $|z|$ denotes Euclidean norm in $\mathbb{C}^m$. Write $|F|_r = \sup_{z \in \mathbb{C}^m,\ |z| \leq r} |F(z)|$ and define $\tau(F)$ the order of exponential type of $F(z)$ by* $\tau(F) = \limsup_{r \to +\infty} \dfrac{\log |F|_r}{r}$.

We then quote a theorem due to Gramain :

Theorem C (Gramain) *Let K be an algebaic number field of degree $[K:\mathbb{Q}] = d$ and $\mathcal{O}_K$ be the integer ring of K. Let $F(z)$ be an entire function in $\mathbb{C}^m$ of $\tau(F) \leq \alpha$. Suppose $F(\mathbb{N}^m) \subset \mathcal{O}_K$. Assume further there exists a constant $c > 0$ such that for $n \in \mathbb{N}^m$ we have*

$$\limsup_{|n| \to \infty} \frac{\log |\overline{F(n)}|}{|n|} \leq c.$$

Then under the condition $\log(e^\alpha - 1) < -(\delta - 1)\log(1 + e^c)$ the function F is a polynomial with coefficients in K.

We collect the results as follows by Pólya, G. H. Hardy, Pisot, Gramain, V. Avanissian & R. Gay, A. Baker and A. Martineau. Below assume the function $F(z)$ is entire.

(1) $F(\mathbb{N}) \subset \mathbb{Z}$, $\tau(F) < \log 2 \Longrightarrow F(z)$ is a polynomial.

(2) $F(\mathbb{Z}) \subset \mathbb{Z}$, $\tau(F) < \log \frac{3+\sqrt{5}}{2} \Longrightarrow F(z)$ is a polynomial.

(3) $F(\mathbb{N}) \subset \mathcal{O}_K$, $\log(e^\alpha - 1) < -(\delta - 1)\log(1 + e^c) \Longrightarrow F(z)$ is a polynomial.

(4) $F(\mathbb{Z}) \subset \mathcal{O}_K$, $\log(2\sinh(\frac{\alpha}{2})) < -\frac{\delta-1}{2}\log(2 + e^c + e^{-c}) \Longrightarrow F(z)$ is a polynomial.

(5) $F(\mathbb{N}^m) \subset Z$, $\tau(F) < \log 2 \Longrightarrow F(z)$ is a polynomial.

(6) $F(\mathbb{Z}^m) \subset Z$, $\tau(F) < \log \frac{3+\sqrt{5}}{2} \Longrightarrow F(z)$ is a polynomial.

(7) $F(\mathbb{N}^m) \subset \mathcal{O}_K$, $\log(e^\alpha - 1) < -(\delta - 1)\log(1 + e^c) \Longrightarrow F(z)$ is a polynomial.

(8) $F(\mathbb{Z}^m) \subset \mathcal{O}_K$, $\log(2\sinh(\frac{\alpha}{2})) < -\frac{\delta-1}{2}\log(2 + e^c + e^{-c}) \Longrightarrow F(z)$ is a polynomial.

The upper bounds $\log 2$ and $\log \frac{3+\sqrt{5}}{2}$ are only optimal.

We get the following in the case where we consider several variables and algebraic integers :

Theorem 6 *Let K be an algebaic number field of degree $[K:\mathbb{Q}] = d$ and $\mathcal{O}_K$ be the integer ring of K. Let L and N_0 be rational integers with $1 < N_0 < L$. Let $m \in \mathbb{N}$. Then there are constants $C_1 > 0$ and $C_2 > 0$ depending on L, N_0, m satisfying the following. Let $\zeta_1, \zeta_2, \cdots$ be infinite points in $\mathbb{C}^m$ pairwise distinct. Let $f_1, \cdots, f_L$ be entire functions in $\mathbb{C}^m$ with $f_j(\zeta_n) \in \mathcal{O}_K$ $(1 \leq \forall j \leq L, \forall n \geq 1)$. Suppose*

$$\frac{\max\limits_{1 \leq j \leq L} \log |f_j|_{C_1 r(N)}}{N} \leq C_2 \quad \forall N \geq N_0.$$

Assume further that there exist $C_3 > 0$ and $C_4 > 0$ such that

$$\frac{\max\limits_{1 \leq j \leq L} \log |\overline{f_j(\zeta_N)}|}{N} \leq C_3, \quad |\zeta_N| < C_4 N \quad \forall N \geq N_0.$$

Then under the condition $\log(e^{C_2} - 1 + m) < -(\delta - 1)\log(1 + e^{C_4})$ we have that the functions $f_1, \cdots, f_L$ are linearly dependent over K.

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